Unit-V

• Characteristic Equation • Eigen Value • Eigen Vector • Linearly Dependent And Independent Eigen Vector • Properties Of Eigen Values And Eigen Vectors: • Cayley Hamilton Theorem: • Diagonalisation Of A Matrix • Diagonalisation By Orthogonal • Transformation Or Orthogonal Reduction • Quadratic Forms • Nature Of Quadratic Forms: • Rules For Finding Nature Of Quadratic Form Using Principal Subdeterminants:

CHARACTERISTIC EQUATION

Let 'A' be a given matrix. Let λ be a scalar. The equation det [A- λ I]=0 is called the characteristic equation of the matrix A.

1. Find the Characteristic equation of A = [(1 4) (2 3)]

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$ ie. $\lambda^2 - D_1\lambda + D_2 = 0$ Where D_1 = Trace of A & $D_2 = |A|$. Therefore $D_1 = 4$ & $D_2 = -5$ implies that $\lambda^2 - 4\lambda - 5 = 0$.

2. Find the Characteristic equation of $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1 \lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ ∴ $D_1 = 3$ & $D_2 = -1$ & $D_3 = -9$ implies that $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$.

EIGEN VALUE

The values of λ obtained from the characteristic equation |A- λ I|=0 are called the Eigen values of A.

EIGEN VECTOR

Let A be a square matrix of order 'n' and λ be a scalar, X be a non- zero column vector such that $AX = \lambda X$.

The non-zero column vector $X = \begin{pmatrix} x_1 \\ x_2 \\ : \\ . \\ x_n \end{pmatrix}$ which satisfies $[A - \lambda I]X = 0$ is called eigen vector or latent vector.

LINEARLY DEPENDENT AND INDEPENDENT EIGEN VECTOR

Let 'A' be the matrix whose columns are eigen vectors.

(i) If |A|=0 then the eigen vectors are linearly dependent.

(ii) If |A|=!0 then the eigen vectors are linearly independent.

1. Find the eigen values and eigen vectors of

$$A = \begin{pmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-6} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \end{pmatrix}$$

Solution: The Characteristic equation of A is $|A-\lambda I|$ =0 ie., $\lambda^3 - D_1 \lambda^2 + D_2\lambda - D_3 = 0$
Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & D_3 =$|A|$

$\therefore D_1 = 18$ & $D_2 = 45$ & $D_3 = 0$ implies that $\lambda^3 - 18 \lambda^2 + 45\lambda = 0$.
$\lambda = -3, -3, 5$ *are the eigen values*

To find eigen vector : By the definition we have $AX = \lambda X$ ie., $(A-\lambda I)X = 0$

$$=> \begin{pmatrix} \mathbf{-2-\lambda} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1-\lambda} & \mathbf{-6} \\ \mathbf{-1} & \mathbf{-2} & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

CASE (I) : When λ= -3 , Substituting in (1) we get

$x_1 + 2x_2 - 3x_3 = 0$;$2x_1 + 4x_2 - 6x_3 = 0$; $-x_1 - 2x_2 + 3x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{0} = \frac{x_2}{3} = \frac{x_3}{2} = k \Rightarrow$ If $\lambda_1 = 0$ then $X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

CASE (ii) : When λ= 5 , Substituting in (1) we get

$-7x_1 + 2x_2 - 3x_3 = 0$; $2x_1 - 4x_2 - 6x_3 = 0$; $-x_1 - 2x_2 - 5x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k \Rightarrow$ If $\lambda_2 = 0$ then $X_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

The eigen vectors are $x_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}; x_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}; x_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Since the eigen values are repeated the eigen vectors are linearly dependent.

2. Find the eigen values and eigen vectors of $A = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1 \lambda^2 + D_2 \lambda - D_3 = 0$. Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 0$ & $D_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -3$ & $D_3 = |A| = 2$ implies that $\lambda^3 - 3\lambda - 2 = 0$

$\lambda = -1, -1, 2$ *are the eigen values*

To find eigen vector : By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} \mathbf{0} - \lambda & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} - \lambda & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

CASE (I) : When $\lambda = -1$, Substituting in (1) we get $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

All the three equations reduce to one and the same equation $x_1 + x_2 + x_3 = 0$

$\therefore$ Two of the unknowns, say x_1 and x_2 are to be treated as free variables. Taking $x_1 = 1$ and $x_2 = 0$, we get $x_3 = -1$ and taking $x_1 = 0$ and $x_2 = 1$, we get $x_3 = -1$.

$$\therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

CASE (ii) : When $\lambda = 2$, Substituting in (1) we get $\begin{pmatrix} \mathbf{-2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-2x_1 + x_2 + x_3 = 0;\ \ x_1 - 2x_2 + x_3 = 0;\ x_1 + x_2 - 2x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ *and* $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Though two of the Eigen values are equal, the Eigen vectors are X1,X2,X3 are linearly independent.

NOTE:

(i) The Eigen vector corresponding to an Eigen value is not unique.

(ii) If all the Eigen values of a matrix are distinct, then the corresponding Eigen vectors are linearly independent.

(iii) If two or more Eigen values are equal, then the Eigen vectors may be linearly independent or linearly dependent.

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS:

Property 1: (I) The sum of the Eigen values of a matrix is equal to the sum of the elements of the principal diagonal (trace of the matrix). i.e., $\lambda_1+ \lambda_2+ \lambda_3=a_{11}+a_{22}+a_{33}$

(ii)The product of the Eigen values of a matrix is equal to the determinant of the matrix. i.e., $\lambda_1 \lambda_2 \lambda_3=|A|$

Property 2: A square matrix A and its transpose ꝏ have the same Eigen values.

Property 3: The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Property 4: If λ is an Eigen value of a matrix A, then $1/ \lambda$, $(\lambda=!0)$ is the Eigen value of A^{-1}

Property 5: If λ is an Eigen value of an orthogonal matrix A, then $1/ \lambda$, $(\lambda=!0)$ is also its Eigen value.

Property 6: If $\lambda_1,\lambda_2,\lambda_3,\dots\lambda_n$ are the Eigen values of a matrix A, then A^m has the Eigen values $\lambda_1^m,\lambda_2^m,\lambda_3^m,\dots.\lambda_n^m$ (m being a positive integer)

Property7: If $\lambda_1,\lambda_2,\lambda_3,\dots\lambda_n$ are the Eigen values of a matrix A, then $k\lambda_1,k\lambda_2,k\lambda_3,\dots k\lambda_n$ are the Eigen values of the matrix KA.

Property 8: Property7: If $\lambda_1,\lambda_2,\lambda_3,\dots\lambda_n$ are the Eigen values of a matrix A and if K is a scalar then $\lambda_1-K,\lambda_2-K,\lambda_3-k,\dots\lambda_n-k$ are the Eigen values of the matrix A-KI.

Property 9: The Eigen values of a real symmetric matrix are real numbers.

Property 10: The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal.

Property 11: The similar matrices have same Eigen values.

Property 12: Eigen vectors of a symmetric matrix corresponding to different Eigen values are orthogonal.

Property 13: If A and B are nxn matrices and B is a non singular matrix then A and $B^{-1}AB$ have same Eigen values.

Property 14: Two Eigen vectors X_1 and X_2 are called orthogonal vectors if $X_1^TX_2=0$

Property 15: If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ be distinct Eigen values of a $n \times n$ matrix then corresponding Eigen vectors $X_1, X_2, X_3, X_4, \ldots X_n$ form a linearly independent set.

Note: The absolute value of a determinant (|detA|) is the product of the absolute values of the eigen values of matrix A.

c = 0 is an eigen value of A if A is a **singular (noninvertible) matrix**

· If A is a n x n **triangular matrix** (upper triangular, lower triangular) or **diagonal matrix**, the eigen values of A are the diagonal entries of A.

· A and its **transpose matrix** have same eigen values.

· Eigen values of a **symmetric matrix** are all real.

· Eigen vectors of a **symmetric matrix** are orthogonal, but only for distinct eigen values.

· The **dominant or principal Eigen** vector of a matrix is an eigen vector corresponding to the Eigen value of largest magnitude (for real numbers, largest absolute value) of that matrix.

· For a **transition matrix**, the dominant Eigen value is always 1.

· The **smallest Eigen value** of matrix A is the same as the inverse (reciprocal) of the largest eigen value of A-1; i.e. of the inverse of A.

1. **Find the Sum and the product of the Eigen values of** $A = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{5} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{pmatrix}$

   Solution: From the property of Eigen values i.e. $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ & $\lambda_1\lambda_2\lambda_3 = |A|$

   Therefore Sum of the Eigen values =1+5+1= 7 & Product of the Eigen values = $|A|$

   i.e., $\lambda_1\lambda_2\lambda_3 =$ 1(5-1)-1(1-3) +5(1-15) = 4+2-70 =-64. Therefore $\lambda_1 + \lambda_2 + \lambda_3 = 7$ & $\lambda_1\lambda_2\lambda_3 = -64$

2. **If** $A = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix}$ **write down the sum and product of the Eigen values of A.**

   Solution: From the property of Eigen values i.e. $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ & $\lambda_1\lambda_2\lambda_3 = |A|$

   Therefore Sum of the Eigen values =1+2+3= 6 & Product of the Eigen values = $|A|$

   Ie, $\lambda_1\lambda_2\lambda_3 =$ 1(6-4)-1(3-2) +1(2-2) = 2-1 =1. Therefore $\lambda_1 + \lambda_2 + \lambda_3 = 6$ & $\lambda_1\lambda_2\lambda_3 = 1$

3. **Find the Sum and the product of the Eigen values of** $A = \begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{pmatrix}$

   Solution: From the property of Eigen values i.e. $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ & $\lambda_1\lambda_2\lambda_3 = |A|$

   Therefore Sum of the Eigen values =6 & Product of the Eigen values = $|A|$ =6

4. **Prove that the Eigen values of $-3A^{-1}$ are the same as those of A = $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{pmatrix}$**

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$

ie. $\lambda^2 - D_1\lambda + D_2 = 0$ Where D_1 = Trace of A & $D_2 = |A|$

Therefore $D_1 = 2$ & $D_2 = -3$ implies that $\lambda^2 - 2\lambda - 3 = 0$. Eigen values of A are 3 and -1.

(By the property of Eigen values we know that if λ_1 & λ_2 are eigen values A then $\frac{1}{\lambda_1}$ & $\frac{1}{\lambda_2}$ are eigen values of A^{-1} also if $-k\lambda_1$&$-k\lambda_2$ are eigen values -kA)

Since Eigen values of A are 3 and -1. Eigen values of A^{-1} are $\frac{1}{3}$ & -1

Therefore Eigen values of $-3A^{-1}$ are -1 and 3.

5. **If the Sum of the two eigen values and trace of 3×3 matrix A are equal. Find the value of $|A|$.**

Solution: Let $\lambda_1, \lambda_2,$ & λ_3 be the eigen values of A. From the property of Eigen values we know that $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ & $\lambda_1\lambda_2\lambda_3 = |A|$. Given $\lambda_1 + \lambda_2 = a_{11} + a_{22} + a_{33}$ that is $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + \lambda_2$ implies that $\lambda_3 = 0$

Therefore Product of the Eigen values = $|A| => \lambda_1\lambda_2\lambda_3 = 0 => |A| = 0$

6. **Prove that if X is an eigen vector of A corresponding to the eigen value λ. Then for any nonzero scalar multiple of A, X is an eigen vector.**

Solution: By definition of eigen values $AX_r = \lambda_r X_r \rightarrow (1)$ Pre multiplying by k on both sides of (1) $(kA)X_r = (k\lambda_r)X_r \;\; => \;\; k\lambda_r$ is the Eigen values of (kA) & X_r is the Eigen vector of (kA).

7. **Two eigen values of a matrix A =** $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ **are equal to 1 each. Find the eigen values of A &** A^{-1}.

Solution: From the property of Eigen values we know that $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$. Given $\lambda_1 = \lambda_2 = 1 \therefore 2 + \lambda_3 = 2 + 3 + 2 = 7 => \lambda_3 = 5$. Therefore the Eigen values of A are 1, 1, 5. (By the property of Eigen values we know that if λ_1& λ_2 are eigen values A then $\frac{1}{\lambda_1}$ & $\frac{1}{\lambda_2}$ are eigen values of A^{-1}) Therefore the Eigen values of A^{-1} are 1, 1, $\frac{1}{5}$.

8. **Find the eigen value of A =** $\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ **corresponding to the eigen vector** $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution: By the definition we have $AX = \lambda X$ ie. $(A - \lambda I)X = 0 => \begin{pmatrix} 2-\lambda & 3 \\ 0 & 4-\lambda \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

2-λ =0 => 2 = λ_1 .BY property $\lambda_1 + \lambda_2 = a_{11} + a_{22}$ => $\lambda_1 + \lambda_2$ = 6 => $\lambda_2 = 4$. Therefore eigen values of A are 2 and 6.

9. **If A is an orthogonal matrix. Show that** A^{-1} **is also orthogonal matrix.**

Solution: Since A is an orthogonal matrix $A^{-1} = A^T$.Implies that (AA^T= A^TA =I) Let B = A^{-1}, to prove B is orthogonal we have to check BB^T= $B^TB = I$.

To prove: BB^T=$A^{-1}(A^{-1})^T = A^T(A^T)^T$=$A^TA$ =I, Since $(A^T)^T$=A.

10. **Find the constants a and b such that the matrix** $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ **has 3 and -2 as its eigen values.**

Solution: BY property $\lambda_1 + \lambda_2 = a_{11} + a_{22}$ & $\lambda_1\lambda_2 = |A|$ => $\lambda_1 + \lambda_2 = a + b$ & $\lambda_1\lambda_2 = ab - 4$ Given $\lambda_1 = 3$ & $\lambda_2 = -2$ => $a + b$ = 1 => b = 1 − a & -6= $ab - 4$ => $ab = -2$ Therefore a(1-a)=-2 => $a^2 - a - 2 = 0$ => a = 2 & a = -1. => b = -1 & b = 2. Therefore when a=-1 then b=2 and when a = 2 then b = -1.

11. **Given that** $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ **verify that eigen values of** A^2 **are the squares of those of A.**

Solution: The Characteristic equation of A is $|A - \lambda I|$ =0

i.e., $\lambda^2 - D_1\lambda + D_2 = 0$ Where D_1 = Trace of A & D_2 = $|A|$. Therefore D_1 = 7 & D_2 = 6 implies that $\lambda^2 - 7\lambda + 6 = 0$. Eigen values are 1 & 6.

$A^2 = \begin{pmatrix} 29 & 28 \\ 7 & 8 \end{pmatrix}$ $\therefore$ The Characteristic equation of A^2 is $\begin{vmatrix} 29-\lambda & 28 \\ 7 & 8-\lambda \end{vmatrix}$ =0

$\lambda^2 - 37\lambda + 36 = 0$. $\therefore$ Eigen values of A^2 are 1 and 36, that are the squares of the eigen values of A, namely 1 and 6.

12. The product of two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 14. Find the third eigen value.

Solution: By the property $\lambda_1\lambda_2\lambda_3 = |A|$. Given that $\lambda_1\lambda_2 = 14$ $\therefore$ $\lambda_3 = \frac{|A|}{\lambda_1\lambda_2} = \frac{28}{14} = 2.$

13. Find the eigen values of $2A^2$, $if\ A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^2 - D_1\lambda + D_2 = 0$ Where D_1 = Trace of A & $D_2 = |A|$. Therefore $D_1 = 6$ & $D_2 = 5$ implies that $\lambda^2 - 6\lambda + 5 = 0.$ Eigen values are 1 & 5.

By the property ${\lambda_1}^2$ *and* ${\lambda_2}^2$ *are the eigen values of* A^2 .

(i.e.) 1 & 25 are the eigen values of A^2.

$\therefore$ The eigen values of $2A^2$ are 2(1) & 2(25) = 2, 50.

14. **Find the sum of the squares of the eigen values of $= \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.**

Solution: By the property "The eigen values of a upper or lower triangular matrix are the main diagonal elements".

Eigen values of A = 3, 2, 5.

Sum of the squares of the eigen values of A =9+4+25 =38.

15. **Find the sum of the eigen values of the inverse of $= \begin{pmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{pmatrix}$.**

Solution: By the property "The eigen values of a upper or lower triangular matrix are the main diagonal elements".

Eigen values of A = 3, 4, 5.

By the property $\frac{1}{\lambda}$ *is the eigen value of* A^{-1} .

$\therefore$ *Eigen value of* $A^{-1} = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

Sum of the eigen values of $A^{-1} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$.

CAYLEY HAMILTON THEOREM:

Every square matrix satisfies its own characteristic equation.

This means that, if $c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda \ + c_n = 0$ is the characteristic equation of a square matrix A of order n, $c_0A^n + c_1A^{n-1} + c_2A^{n-2} + \cdots + c_{n-1}A + c_nI = 0$, where I is the unit matrix of order n.

1. If A is non singular matrix then we can get A-1, using this theorem

$$A^{-1} = \frac{-1}{a_0}[a_1I + a_2A + a_3A^2 + \cdots .. + a_nA^{n-1}]$$

2. Higher positive integral powers of A can be computed

$$A^{n+1} = \frac{-1}{a_0}[a_0A + a_1A^2 + a_2A^3 + \cdots .. + a_{n-1}A^n]$$

1. **Verify the Cayley Hamilton theorem for the matrix A =** $\begin{pmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{5} \end{pmatrix}$

Solution: Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem. The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^2 - D_1\lambda + D_2 = 0$ Where D_1 = Trace of A & $D_2 = |A|$

Therefore $D_1 = 8$ & $D_2 = 14$ implies that $\lambda^2 - 8\lambda + 14 = 0$ We have to check $A^2 - 8A + 14I = 0$

$A^2 = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 10 & -8 \\ -8 & 26 \end{pmatrix}$ & $8A = \begin{pmatrix} 24 & -8 \\ -8 & 40 \end{pmatrix}$

L.H.S = $A^2 - 8A + 14I$

$= \begin{pmatrix} 10 & -8 \\ -8 & 26 \end{pmatrix} - \begin{pmatrix} 24 & -8 \\ -8 & 40 \end{pmatrix} + \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ = R.H.S => Cayley Hamilton theorem is verified.

2. **Using Cayley Hamilton theorem find** A^{-1} **given A =** $\begin{pmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{pmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$ i.e., $\lambda^2 - D_1\lambda + D_2 = 0$ Where D_1 = Trace of A & $D_2 = |A|$. Therefore $D_1 = 4$ & $D_2 = -5$ implies that $\lambda^2 - 4\lambda - 5 = 0$. By Cayley Hamilton theorem we have $A^2 - 4A - 5I = 0$ .....(1) , Premultiplying by A^{-1} on both sides of (1) we get $A^{-1} = \frac{1}{5}\ [A - 4I]$ => $A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ -2 & -1 \end{pmatrix}$

3. **Using Cayley Hamilton theorem find** A^{-1} **for A =** $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ $\therefore D_1 = 3$ & $D_2 = -1$ & $D_3 = -9$ implies that $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$.

(Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem.) By Cayley Hamilton theorem we have $A^3 - 3A^2 - A + 9I = 0$ .....(1) Premultiplying by A^{-1} on both sides of (1) we get $A^{-1} = \frac{1}{9}\ [-A^2 + 3A + I]$

$$=> \quad A^{-1} = \frac{1}{9}\left[\begin{pmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$=> \quad A^{-1} = \frac{1}{9}\left[\begin{pmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{pmatrix}\right]$$

4. Using Cayley Hamilton theorem find A^{-1} for A = $\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

SOLUTION : The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ $\therefore D_1 = 5$ & $D_2 = 9$ & $D_3 = 1$ implies that $\lambda^3 - 5\,\lambda^2 + 9\lambda - 1 = 0$. (Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem.) By Cayley Hamilton theorem we have

$A^3 - 5\,A^2 + 9A - I = 0$ .....(1) , Premultiplying by A^{-1} on both sides of (1)

we get $A^{-1} = [A^2 - 5A + 9I]$

$$=> \quad A^{-1} = \left[\begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 10 & -10 \\ -5 & 15 & 0 \\ 0 & -10 & 5 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right]$$

$$=> \quad A^{-1} = \left[\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}\right]$$

5. Using Cayley Hamilton theorem find A^{-1} for A = $\begin{pmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{pmatrix}$

SOLUTION : The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ $\therefore D_1 = 8$ & $D_2 = 8$ & $D_3 = 1$ implies that $\lambda^3 - 8\,\lambda^2 + 8\lambda - 1 = 0$ (Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem.) By Cayley Hamilton theorem we have $A^3 - A^2 + 8A - I = 0$ .....(1)

Premultiplying by A^{-1} on both sides of (1) we get $A^{-1} = [A^2 - 8A + 8I]$

$$=> \quad A^{-1} = \left[\begin{pmatrix} -8 & 0 & 21 \\ 21 & 1 & -39 \\ -21 & 0 & 55 \end{pmatrix} - \begin{pmatrix} -8 & 0 & 24 \\ 64 & 8 & -56 \\ -24 & 0 & 64 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right]$$

$$=> \quad A^{-1} = \left[\begin{pmatrix} 8 & 0 & -3 \\ -43 & 1 & 17 \\ 3 & 0 & -1 \end{pmatrix}\right]$$

6. **Verify Cayley Hamilton theorem and also find A^5 interms of A^2 , A & I of A =** $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix}$

SOLUTION : The Characteristic equation of A is $|A-\lambda I|=0$ ie., $\lambda^3 - D_1\ \lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ $\therefore D_1 = 5$ & $D_2 = 7$ & $D_3 = 3$ implies that $\lambda^3 - 5\,\lambda^2 + 7\lambda - 3 = 0$

(Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem.)

To verify C.H.T we have check : $A^3 - 5\,A^2 + 7A - 3I = 0 \ \ldots\ldots (i)$

Consider L.H.S of (I) : $A^3 - 5\,A^2 + 7A - 3I$

$$= \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix} - \begin{pmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{pmatrix} + \begin{pmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \text{R.H.S of (i)}$$

Therefore C.H.T is verified.

By Cayley Hamilton theorem we have $A^3 - 5\,A^2 + 7A - 3I = 0 \quad \ldots(1)$,

$$=> A^3 = [5A^2 - 7A + 3I]$$

$$= \left[\begin{pmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{pmatrix} - \begin{pmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix}$$

Premultiplying by A^1 on both sides of (1) we get

$$=> A^4 = [5A^3 - 7A^2 + 3A^1] \ \ldots\ldots\ldots(2)$$

$$= 5\begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix} - 7\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + 3\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{pmatrix}$$

Premultiplying by A^1 on both sides of (2) we get

$$=> A^5 = [5A^4 - 7A^3 + 3A^2]$$

$$=> A^5 = \left[\begin{pmatrix} 205 & 200 & 200 \\ 0 & 5 & 0 \\ 200 & 200 & 205 \end{pmatrix} - \begin{pmatrix} 98 & 91 & 91 \\ 0 & 7 & 0 \\ 91 & 91 & 98 \end{pmatrix} + \begin{pmatrix} 15 & 12 & 12 \\ 0 & 3 & 0 \\ 12 & 12 & 15 \end{pmatrix}\right]$$

$$=> A^5 = \left[\begin{pmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{pmatrix}\right]$$

7. **Verify Cayley Hamilton theorem and also find A^4 interms of A^2 , A & I of A =** $\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$

SOLUTION : The Characteristic equation of A is $|A-\lambda I| = 0$ ie., $\lambda^3 - D_1\ \lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ $\therefore D_1 = 6$ & $D_2 = 8$ & $D_3 = 3$ implies that $\lambda^3 - 6\,\lambda^2 + 8\lambda - 3 = 0$

(Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem.)

To verify C.H.T we have check : $A^3 - 6\,A^2 + 8A - 3I = 0$ ... . (i)

Consider L.H.S of (I) : $A^3 - 6\,A^2 + 8A - 3I$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - \begin{pmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{pmatrix} + \begin{pmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \text{R.H.S of (i)}$$

Therefore C.H.T is verified.

By Cayley Hamilton theorem we have $A^3 - 6\,A^2 + 8A - 3I = 0$ .....(1) ,

$A^3 = 6\,A^2 - 8A + 3I.$

$\Rightarrow A^4 = [6A^3 - 8A^2 + 3A]$

$= 6\,(6A^2 - 8A + 3I) + [-8A^2 + 3A]$

$= 28A^2 - 45A + 18I$

$$\Rightarrow A^4 = \left[28\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - \begin{pmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{pmatrix} + \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}\right]$$

$$\therefore A^4 = \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

8. **Verify the Cayley Hamilton Theorem and hence find A^{-1} for $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$**

Ans: : The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ $\therefore D_1 = 6$ & D_2 = -9 & $D_3 = 4 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.

By Cayley Hamilton theorem we have $A^3 - 6A^2 + 9A - 4I = 0$ .....(1)

Premultiplying by A^{-1} on both sides of (1) we get $A^{-1} = \frac{1}{4}[A^2 - 6A + 9I]$

$$=> \quad A^{-1} = \frac{1}{4}\left[\begin{pmatrix} -6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} + \begin{pmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}\right] = \frac{1}{4}\left[\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}\right]$$

DIAGONALISATION OF A MATRIX

The process of finding a matrix M such that M-1AM=D ,where D is a diagonal matrix, if called diagonalisation of the Matrix A

Note Ak=MDkM-1

DIAGONALISATION BY ORTHOGONAL

TRANSFORMATION OR ORTHOGONAL REDUCTION

If A is a real symmetric matrix, then the Eigen vectors of A will be not only linearly independent but also pair wise orthogonal. If we normalize each eigen vector Xr i.e. divide each element of Xr by the square root of the sum of the square\s of all the elements of Xr and use the normalized Eigen vectors of A to form the normalized modal matrix N, then it can be proved that N is an orthogonal matrix. By a property of orthogonal matrix, N-1= NT.

The similarity transformation M-1AM=D takes the form NTAN=D

Transforming A into D by means of the transformation NTAN=D is known as orthogonal transformation or orthogonal reduction.

NOTE:- Diagonalisation by orthogonal transformation is possible only for a real symmetric matrix.

1. Diagonalise the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ by an orthogonal transformation.

SOLUTION: Given $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1 \lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 12$ & $D_2 = 36$ & $D_3 = 32$ implies that $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

$\therefore$ *The e*igen values of the matrix A are 2 , 2 & 8.

To find eigen vector : By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=8 , Substituting in (2) we get

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

Solving using cross multiplication rule $\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} = k$ => If $\lambda_1 = 8$ then $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

CASE(ii) : When λ=2 , Substituting in (2) we get

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

We have only one equation $2x_1 - x_2 + x_3 = 0$ with three unknowns, let $x_2 = 2x_1 + x_3$

if $x_1 = 0, x_3 = 1$ *then* $x_2 = 1$ => If $\lambda_2 = 2$ then $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=2 , Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ From orthogonal transformation we know that X_1, X_2& X_3 must be mutually perpendicular to each other. => $X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$

$$2a - b + c = 0$$

$$0a + b + c = 0$$

Solving using cross multiplication rule $\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2} = k$ => If $\lambda_3 = 2$ then $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\therefore$ Modal matrix M= $\begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$$

2. Diagonalise the matrix $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by an orthogonal transformation.

Solution: Given A = $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 9$ & $D_2 = 24$ & $D_3 = 16$ implies that $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$.

$\therefore$ *The* eigen values of the matrix A are 4 , 4 & 1.

To find eigen vector: By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=1 , Substituting in (2) we get

$2x_1 + x_2 + x_3 = 0$

$x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 + 2x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{-3} = k$ => If $\lambda_1 = 1$ then $X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

CASE(ii) : When λ=4 , Substituting in (2) we get

$-x_1 + x_2 + x_3 = 0$

$x_1 - x_2 - x_3 = 0$

$x_1 - x_2 - x_3 = 0$

$x_1 - x_2 - x_3 = 0$

We have only one equation $x_1 - x_2 - x_3 = 0$ with three unknowns, let $x_1 = x_2 + x_3$ if $x_1 = 0, x_3 = 1$ *then* $x_2 = -1$ => If $\lambda_2 = 4$ then $X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=4 , Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ From orthogonal transformation we know that X_1, X_2& X_3 must be mutually perpendicular to each other. => $X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$

$-a + b + c = 0$

$0a - b + c = 0$

Solving using cross multiplication rule $\frac{a}{2} = \frac{b}{1} = \frac{c}{1} = k$ => If $\lambda_3 = 4$ then $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\therefore$ Modal matrix M= $\begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D$$

3. Diagonalise the matrix $\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$ by an orthogonal transformation.

Solution: Given A = $\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1 \lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 17$ & $D_2 = 42$ & $D_3 = 0$ implies that $\lambda^3 - 17\lambda^2 + 42\lambda = 0$.

$\therefore$ *The e*igen values of the matrix A are 0 , 3 & 14.

To find eigen vector: By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=0 , Substituting in (2) we get

$10x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 + 2x_2 + 3x_3 = 0$

$-5x_1 + 3x_2 + 5x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4} = k$ => If $\lambda_1 = 0$ then $X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

CASE(ii) : When λ=3 , Substituting in (2) we get

$7x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 - x_2 + 3x_3 = 0$

$-5x_1 + 3x_2 + 2x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11} = k$ => If $\lambda_2 = 3$ then $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=14 , Substituting in (2) we get

$-4x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 - 12x_2 + 3x_3 = 0$

$-5x_1 + 3x_2 - 9x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{99} = \frac{x_2}{-33} = \frac{x_3}{-66} = k$ => If $\lambda_3 = 14$ then $X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

∴ Modal matrix M= $\begin{pmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix}$

Normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} = D$$

QUADRATIC FORMS

A homogeneous polynomial of the second degree in any number of variables is called a quadratic form.

For example, x12+2x22-3x32+5x1x2-6x1x3+4x2x3 is a quadratic form in three variables.

The symmetric matrix

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$$

is called the matrix of the quadratic form Q.

NOTE:

To find the symmetric matrix A of a quadratic form, the coefficient of x_i^2 is placed in the a_{ii} position and $\left(\frac{1}{2} \times coefficient\ of\, x_i x_j\right)$ is placed in each of the a_{ij} and a_{ji} positions.

CANONICAL FORM OF A QUADRATIC FORM: Let $Q = X^T AX$ be a quadratic form in n variables $x_1, x_2, \ldots x_n$.

In the linear transformation X = PY, if P is chosen such that $B = P^T AP$ is a diagonal matrix of the form $\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$, then the quadratic form Q gets reduced as Q= $Y^T BY$

$$= [y_1 \quad y_2 \cdots \quad y_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

This form of Q is called the sum of the squares form of Q or the canonical form of Q.

NATURE OF QUADRATIC FORMS:

Rank: When the quadratic form is reduced to the canonical form it contains only r terms which is the rank of A.

Index: The number of positive terms in the canonical form is called the index (p) of the quadratic form.

Signature: The difference between the number of positive and negative terms is called signature (s) of the quadratic form [s = 2p-r].

The quadratic form Q=XTAX in n variables is said to be

(i) Positive definite: If r = n and p = n or if all the Eigen values of A are positive.

(ii) Positive semi definite: If r < n and p = r or if all the Eigen values of A>=0 and atleast one Eigen value is zero.

(iii) Negative definite: If r = n and p = 0 or if all the Eigen values of A are negative.

(iv) Negative semi definite: If r < n and p = 0 or if all the Eigen values of A<=0 and atleast one Eigen value is zero.

(v) Indefinite: In all other cases or if A has positive as well as negative Eigen values.

RULES FOR FINDING NATURE OF QUADRATIC FORM USING PRINCIPAL SUBDETERMINANTS:

In this method we can determine the nature of the quadratic form without reducing it to the canonical form. Let A be a square matrix of order n.

$$D_1 = |a_{11}|$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$D_n = |A|$$

Here D1,D2,D3 … … … … Dn are called the principal subdeterminants of A. From D1,D2,D3 … … … … Dn , the nature of the quadratic form can be determined.

1. A Q.F is positive definite if D1,D2,D3 … … … … Dn are all positive i.e., Dn>0 for all n.

2. A Q.F is negative definite if D1,D3,D5 … are all negative and D2,D4,D6 … are all positive i.e.,(-1)n Dn>0 for all n.

3. A Q.F is positive semi- definite if Dn>=0 and atleast one Di=0.

4. A Q.F is negative semi- definite if (-1)nDn >=0 and atleast one Di=0.

5. A Q.F is indefinite in all other cases.

1. Without reducing to canonical form find the nature of the Quadratic form $x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

Solution: Matrix of the Quadratic form is A = $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

$D_1 = 1 > 0$, $D_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$ & $D_3 = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = 0 -2-2 =-4$

Since $D_1 > 0$, $D_2 = 0$ & $D_3 < 0$ $\therefore$ Nature of the Quadratic form is indefinite.

2. **Reduce the quadratic form $x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

Matrix form of Quadratic form = $X^T AX \rightarrow (1)$ where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1 \lambda^2 + D_2 \lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 3$ & $D_2 = 0$ & $D_3 = -4$ implies that $\lambda^3 - 3\lambda^2 + 4 = 0$.

$\therefore$ *The* eigen values of the matrix A are -1, 2 & 2.

To find Eigen vector : By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=-1 , Substituting in (2) we get

$2x_1 - x_2 - x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$-x_1 - x_2 + 2x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} = k$ => If $\lambda_1 = -1$ then $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

CASE(ii) : When λ=2 , Substituting in (2) we get

$-x_1 - x_2 - x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

We have only one equation $x_1 + x_2 + x_3 = 0$ with three unknowns, let $x_1 = -(x_2 + x_3)$

if $x_2 = 0, x_3 = 1$ *then* $x_1 = -1$ => If $\lambda_2 = 2$ then $X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

CASE (iii) : When λ=2 , Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ From orthogonal transformation we know that $X_1, X_2 \& X_3$

must be mutually perpendicular to each other. => $X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$

$a + b + c = 0$

$-a - 0b + c = 0$

Solving using cross multiplication rule $\frac{a}{1} = \frac{b}{-2} = \frac{c}{1} = k$ => If $\lambda_3 = 2$ then $X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\therefore$ Modal matrix M= $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}$

Normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T AX$

$= [(NY)^T A(NY)]$

$= Y^T [N^T AN] Y$

$= Y^T DY$

$$= (y_1 \quad y_2 \quad y_3) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Q.F $= -y_1^2 + 2y_2^2 + 2y_3^2$ which is the canonical form of the quadratic form

Nature of the Q.F = Indefinite

Rank of the Q.F (r) = 3

Index of the Q.F (p) = 2

Signature of the Q.F (s) =2p-r = 1.

3. **Reduce the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4xz$ to canonical form using orthogonal transformation also find its nature, rank, index & signature.**

Solution: Quadratic form = $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4xz$

Matrix form of Quadratic form = $X^T AX \rightarrow (1)$ where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 18$ & $D_2 = 45$ & $D_3 = 0$ implies that $\lambda^3 - 18\,\lambda^2 + 45\lambda = 0$.

$\therefore$ *The* eigen values of the matrix A are 0, 3 & 15.

To find eigen vector: By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=0 , Substituting in (2) we get

$8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} = k$ => If $\lambda_1 = 0$ then $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

CASE(ii) : When λ=3 , Substituting in (2) we get

$5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 0x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} = k$ => If $\lambda_2 = 3$ then $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

CASE(iii) : When λ=15 , Substituting in (2) we get

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} = k$ => If $\lambda_3 = 15$ then $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Since $X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$ => X_1, X_2& X_3 are mutually perpendicular to each other.

$\therefore$ Modal matrix M= $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

Normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{9}} & \frac{2}{\sqrt{9}} & \frac{2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{1}{\sqrt{9}} & \frac{-2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{-2}{\sqrt{9}} & \frac{1}{\sqrt{9}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{9}} & \frac{2}{\sqrt{9}} & \frac{2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{1}{\sqrt{9}} & \frac{-2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{-2}{\sqrt{9}} & \frac{1}{\sqrt{9}} \end{pmatrix} \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{9}} & \frac{2}{\sqrt{9}} & \frac{2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{1}{\sqrt{9}} & \frac{-2}{\sqrt{9}} \\ \frac{2}{\sqrt{9}} & \frac{-2}{\sqrt{9}} & \frac{1}{\sqrt{9}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T AX$

$= [(NY)^T A(NY)]$

$= Y^T [N^T AN] Y$

$= Y^T DY$

$= (y_1 \quad y_2 \quad y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= 0{y_1}^2 + 3{y_2}^2 + 15{y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = Positive semi definite

Rank of the Q.F (r) = 2

Index of the Q.F (p) = 2

Signature of the Q.F (s) =2p-r = 2.

4. **Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$

Matrix form of Quadratic form = $X^T AX \rightarrow (1)$ where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1 \lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 12$ & $D_2 = 36$ & $D_3 = 32$ implies that $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

$\therefore$ *The* eigen values of the matrix A are 8 , 2 & 2.

To find eigen vector : By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$=> \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$

CASE (I) : When λ=8 , Substituting in (2) we get

$-2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} = k$ => If $\lambda_1 = 8$ then $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

CASE(ii) : When λ=2 , Substituting in (2) we get

$4x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

We have only one equation $2x_1 - x_2 + x_3 = 0$ with three unknowns, let $x_2 = 2x_1 + x_3$

if $x_1 = 0, x_3 = 1$ *then* $x_2 = 1$ => If $\lambda_2 = 2$ then $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=2 , Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ From orthogonal transformation we know that X_1, X_2 & X_3 must be mutually perpendicular to each other. => $X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$

$2a - b + c = 0$

$0a + b + c = 0$

Solving using cross multiplication rule $\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2} = k$ => If $\lambda_3 = 2$ then $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\therefore$ Modal matrix M= $\begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T AX$

$=[(NY)^T A(NY)]$

$=Y^T [N^T AN]Y$

$= Y^T DY$

$=(y_1 \quad y_2 \quad y_3) \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= 8{y_1}^2 + 2{y_2}^2 + 2{y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = Positive definite

Rank of the Q.F (r) = 3

Index of the Q.F (p) = 3

Signature of the Q.F (s) =2p-r = 3.

5. **Reduce the quadratic form $2x^2 + 2y^2 + z^2 + 4xy$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $2x^2 + 2y^2 + z^2 + 4xy$

Matrix form of Quadratic form = $X^T AX \rightarrow (1)$ where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & D_3 =$|A|$

$\therefore D_1 = 5$ & D_2 =4 & $D_3 = 0$ implies that $\lambda^3 - 5\,\lambda^2 + 4\lambda = 0$.

$\therefore$ *The* eigen values of the matrix A are 0, 1 & 4.

To find eigen vector: By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=0 , Substituting in (2) we get

$2x_1 + 2x_2 + 0x_3 = 0$

$2x_1 + 2x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{0} = k$ => If $\lambda_1 = 0$ then $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

CASE(ii) : When λ=1 , Substituting in (2) we get

$x_1 + 2x_2 + 0x_3 = 0$

$2x_1 + x_2 + 0x_3 = 0$

$0x_1 + 0x_2 - 0x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-3} = k$ => If $\lambda_2 = 1$ then $X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

CASE(iii) : When λ=4 , Substituting in (2) we get

$-2x_1 + 2x_2 + 0x_3 = 0$

$2x_1 - 2x_2 + 0x_3 = 0$

$0x_1 + 0x_2 - 3x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{0} = k$ => If $\lambda_3 = 4$ then $X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Since $X_1.X_2^{\,T} = 0$, $X_2.X_3^{\,T} = 0$ & $X_3.X_1^{\,T} = 0$ => X_1, X_2& X_3 are mutually perpendicular to each other.

$\therefore$ Modal matrix M= $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{-\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix}$

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{-\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^TAX$

$=[(NY)^TA(NY)]$

$=Y^T[N^TAN]Y$

$= Y^TDY$

$=(y_1 \quad y_2 \quad y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= 0{y_1}^2 + {y_2}^2 + 4{y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = Positive semi definite

Rank of the Q.F (r) = 2

Index of the Q.F (p) = 2

Signature of the Q.F (s) =2p-r = 2.

6. **Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$

Matrix form of Quadratic form = $X^TAX \rightarrow$ (1) where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 11$ & $D_2 = 36$ & $D_3 = 36$ implies that $\lambda^3 - 11\,\lambda^2 + 36\lambda - 36 = 0$.

$\therefore$ *The* eigen values of the matrix A are 2, 3 & 6.

To find eigen vector : By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=2 , Substituting in (2) we get

$x_1 - x_2 + x_3 = 0$

$-x_1 + 3x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} = k$ => If $\lambda_1 = 2$ then $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

CASE(ii) : When λ=3 , Substituting in (2) we get

$0x_1 - x_2 + x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 + 0x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1} = k$ => If $\lambda_2 = 3$ then $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=6 , Substituting in (2) we get

$-3x_1 - x_2 + x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

$x_1 - x_2 - 3x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2} = k$ => If $\lambda_3 = 15$ then $X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$Since\ X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$ => X_1, X_2& X_3 are mutually perpendicular to each other.

∴ Modal matrix M= $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{-\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T AX$

$=[(NY)^T A(NY)]$

$=Y^T [N^T AN]Y$

$= Y^T DY$

$=(y_1 \quad y_2 \quad y_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= 2{y_1}^2 + 3{y_2}^2 + 6{y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = Positive definite

Rank of the Q.F (r) = 3

Index of the Q.F (p) = 3

Signature of the Q.F (s) =2p-r = 3.

7. **Reduce the quadratic form $3x_1{}^2 + 2x_2{}^2 + 3x_3{}^2 - 2x_1x_2 - 2x_2x_3$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $3x_1{}^2 + 2x_2{}^2 + 3x_3{}^2 - 2x_1x_2 - 2x_2x_3$

Matrix form of Quadratic form = $X^T AX \rightarrow (1)$ where X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ & A = $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A, D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 8$ & $D_2 = 19$ & $D_3 = 12$ implies that $\lambda^3 - 8\,\lambda^2 + 19\lambda - 12 = 0$.

$\therefore$ *The* eigen values of the matrix A are 1, 3 & 4.

To find eigen vector: By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=1 , Substituting in (2) we get

$2x_1 - x_2 + 0x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

$0x_1 - x_2 + 2x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k$ => If $\lambda_1 = 1$ then $X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

CASE(ii) : When λ=3 , Substituting in (2) we get

$0x_1 - x_2 + 0x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

$0x_1 - x_2 + 0x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k$ => If $\lambda_2 = 3$ then $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

CASE(iii) : When λ=4 , Substituting in (2) we get

$-x_1 - x_2 + 0x_3 = 0$

$-x_1 - 2x_2 - x_3 = 0$

$0x_1 - x_2 - x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} = k$ => If $\lambda_3 = 4$ then $X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$Since\ X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$ => X_1, X_2& X_3 are mutually perpendicular to e
other.

∴ Modal matrix M= $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T AX$

$=[(NY)^T A(NY)]$

$=Y^T [N^T AN]Y$

$= Y^T DY$

$=(y_1 \quad y_2 \quad y_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= {y_1}^2 + 3{y_2}^2 + 4{y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = Positive definite

Rank of the Q.F (r) = 3

Index of the Q.F (p) = 3

Signature of the Q.F (s) =2p-r = 3.

8. **Reduce the quadratic form $2{x_1}^2 + 6{x_2}^2 + 2{x_3}^2 + 8x_1x_3$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $2{x_1}^2 + 6{x_2}^2 + 2{x_3}^2 + 8x_1x_3$

Matrix form of Quadratic form = $X^T AX \rightarrow (1)$ where X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ & A = $\begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A, D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 10$ & $D_2 = 12$ & $D_3 = 72$ implies that $\lambda^3 - 10\,\lambda^2 + 12\lambda - 72 = 0$.

$\therefore$ *The e*igen values of the matrix A are -2, 6 & 6.

To find eigen vector: By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=-2 , Substituting in (2) we get

$4x_1 + 0x_2 + 4x_3 = 0$

$0x_1 + 8x_2 + 0x_3 = 0$

$4x_1 + 0x_2 + 4x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{-32} = \frac{x_2}{0} = \frac{x_3}{32} = k$ => If $\lambda_1 = -2$ then $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

CASE(ii) : When λ=6 , Substituting in (2) we get

$-4x_1 + 0x_2 + 4x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

$4x_1 + 0x_2 - 4\,x_3 = 0$

We have only one equation $x_1 + 0x_2 - x_3 = 0$ with three unknowns, let $x_1 = x_3$,

if $x_1 = 1, then\ x_3 = 1$ & $x_2 = arbitrary\ ie., x_2 = 0$ => If $\lambda_2 = 6$ then $X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=6 , Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ From orthogonal transformation we know that $X_1, X_2 \& X_3$ must be mutually perpendicular to each other. => $X_1.X_2^T = 0$, $X_2.X_3^T = 0$ & $X_3.X_1^T = 0$

$a + 0b - c = 0$

$a + 0b + c = 0$

Solving using cross multiplication rule $\frac{a}{0} = \frac{b}{-2} = \frac{c}{0} = k$ => If $\lambda_3 = 2$ then $X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

∴ Modal matrix M= $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T AX$

$=[(NY)^T A(NY)]$

$=Y^T [N^T AN]Y$

$= Y^T DY$

$=(y_1 \quad y_2 \quad y_3) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= -2{y_1}^2 + 6{y_2}^2 + 6{y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = In definite

Rank of the Q.F (r) = 3

Index of the Q.F (p) = 2

Signature of the Q.F (s) =2p-r = 1.

9. **Reduce the quadratic form $2x_1x_2 + 2x_1x_3 + 2x_2x_3$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $2x_1x_2 + 2x_1x_3 + 2x_2x_3$

Matrix form of Quadratic form = $X^TAX \rightarrow (1)$ where X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ & A = $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 0$ & $D_2 = 3$ & $D_3 = 2$ implies that $\lambda^3 - 3\lambda - 2 = 0$.

$\therefore$ *The e*igen values of the matrix A are 2, -1 & -1.

To find eigen vector: By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=2 , Substituting in (2) we get

$-2x_1 + x_2 + x_3 = 0$

$x_1 - 2x_2 + x_3 = 0$

$x_1 + x_2 - 2x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} = k$ => If $\lambda_1 = 2$ then $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

CASE(ii) : When λ=-1 , Substituting in (2) we get

$x_1 + x_2 + x_3 = 0$

$x_1 + x_2 + x_3 = 0$

$x_1 + x_2 + x_3 = 0$

We have only one equation $x_1 + x_2 + x_3 = 0$ with three unknowns, let $x_1 = -(x_3 + x_2)$

if $x_2 = 0, x_3 = 1 then\ x_1 = -1$ => If $\lambda_2 = -1$ then $X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=-1 , Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ From orthogonal transformation we know that $X_1, X_2 \& X_3$ must be mutually perpendicular to each other. => $X_1 . X_2^T = 0$, $X_2 . X_3^T = 0$ & $X_3 . X_1^T = 0$

$a + b + c = 0$

$-a + 0b + c = 0$

Solving using cross multiplication rule $\frac{a}{1} = \frac{b}{-2} = \frac{c}{1} = k$ => If $\lambda_3 = -1$ then $X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

∴ Modal matrix M= $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}$

Normalised modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

$$N^T A N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{-\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T A X$

$= [(NY)^T A (NY)]$

$= Y^T [N^T A N] Y$

$= Y^T D Y$

$= (y_1 \quad y_2 \quad y_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= 2{y_1}^2 - {y_2}^2 - {y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = In definite

Rank of the Q.F (r) = 3

Index of the Q.F (p) = 1

Signature of the Q.F (s) =2p-r =- 1.